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PROOF OF THE SOUL CONJECTURE OF CHEEGER AND GROMOLL

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In this note we consider complete noncompact Riemannian manifolds M of nonnegative sectional curvature. The structure of such manifolds was discovered by Cheeger and Gromoll [2]: M contains a (not necessarily unique) totally convex and totally geodesic submanifold S without boundary, $0 \leq \dim S < \dim M$, such that M is diffeomorphic to the total space of the normal bundle of S in M. (S is called a soul of M.) In particular, if S is a single point, then M is diffeomorphic to a Euclidean space. This is the case if all sectional curvatures of M are positive, according to an earlier result of Gromoll and Meyer [3]. Cheeger and Gromoll conjectured that the same conclusion can be obtained under the weaker assumption that M contains a point where all sectional curvatures are positive. A contrapositive version of this conjecture expresses certain rigidity of manifolds with souls of positive dimension. It was verified in [2] in the cases $\dim S = 1$ and $\operatorname{codim} S = 1$, and by Marenich, Walschap, and Strake in the case $\operatorname{codim} S = 2$. Recently Marenich [4] published an argument for analytic manifolds without dimensional restrictions. (We were unable to get through that argument, containing over 50 pages of computations.)

In this note we present a short proof of the Soul Conjecture in full generality. Our argument makes use of two basic results: the Berger's version of Rauch comparison theorem [1] and the existence of distance nonincreasing retraction of M onto S due to Sharafutdinov [5].

Theorem. *Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature, let S be a soul of M, and let $P\colon M \to S$ be a distance nonincreasing retraction.*

(A) *For any $x \in S$, $\nu \in SN(S)$ we have*

$$P(\exp_x(t\nu)) = x \quad \text{for all } t \geq 0,$$

where $SN(S)$ denotes the unit normal bundle of S in M.

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(B) *For any geodesic* $\gamma \subset S$ *and any vector field* $\nu \in \Gamma(SN(S))$ *parallel along* γ, *the "horizontal" curves* γ_t, $\gamma_t(u) = \exp_{\gamma(u)}(t\nu)$, *are geodesics, filling a flat totally geodesic strip* $(t \geq 0)$. *Moreover, if* $\gamma[u_0, u_1]$ *is minimizing, then all* $\gamma_t[u_0, u_1]$ *are also minimizing.*

(C) P *is a Riemannian submersion of class* C^1. *Moreover, the eigenvalues of the second fundamental forms of the fibers of* P *are bounded above, in barrier sense, by* $\text{injrad}(S)^{-1}$.

The Soul Conjecture is an immediate consequence of (B) *since the normal exponential map* $N(S) \to M$ *is surjective.*

Proof. We prove (A) and (B) first. Clearly it is sufficient to check that if (A) and (B) hold for $0 \leq t \leq l$ for some $l \geq 0$, then they continue to hold for $0 \leq t \leq l + \varepsilon(l)$, for some $\varepsilon(l) > 0$. In particular, we can start from $l = 0$, in which case some of the details of the argument below are redundant.

Suppose that (A) and (B) hold for $0 \leq t \leq l$. For small $r \geq 0$ consider a function $f(r) = \max\{|xP(\exp_x((l+r)\nu))| | x \in S, \nu \in SN_x(S)\}$. Clearly f is a Lipschitz nonnegative function, and $f(0) = 0$. We are going to prove that $f \equiv 0$ (thereby establishing (A) for $0 \leq t \leq l + \varepsilon(l)$) by showing that the upper left derivative of f is nowhere positive.

Fix $r > 0$. Let $f(r) = |x_0 - \overline{x}_0|$ where $\overline{x}_0 = P(\exp_{x_0}((l+r)\nu_0)$. Since r is small and P is distance decreasing, we can assume that $|x_0\overline{x}_0| <$ injrad(S). Pick a point $x_1 \in S$ so that x_0 lies on a minimizing geodesic between $\overline{x}_0$ and x_1; let $x_0 = \gamma(u_0)$, $x_1 = \gamma(u_1)$. Let $\nu \in \Gamma(SN(S))$ be a parallel vector field along γ, $\nu|_{x_0} = \nu_0$. Then, according to our assumption, the curves $\gamma_t(u) = \exp_{\gamma(u)}(t\nu)$, $0 \leq t \leq l$, are minimizing geodesics of constant length filling a flat totally geodesic rectangle. In particular, the tangent vectors to the normal geodesics $\sigma_u(t) = \exp_{\gamma(u)}(t\nu)$ form a parallel vector field along γ_l. Therefore, according to Berger's comparison theorem, the arcs of γ_{l+r} are no longer than corresponding arcs of γ_l, with equality only if γ_t, $l \leq t \leq l + r$, are geodesics filling a flat totally geodesic rectangle.

Now consider the point $\overline{x}_1 = P(\sigma_{u_1}(l+r))$. Using the distance decreasing property of P and the above observation we get

$$|\overline{x}_0\overline{x}_1| \leq |\sigma_{u_0}(l+r)\sigma_{u_1}(l+r)| \leq |\sigma_{u_0}(l)\sigma_{u_1}(l)| = |x_0x_1|. \tag{1}$$

On the other hand,

$$|x_1\overline{x}_1| \leq f(r) = |x_0\overline{x}_0|. \tag{2}$$

Taking into account that by construction

$$|x_0\overline{x}_0| + |x_0x_1| = |\overline{x}_0x_1| \leq |x_1\overline{x}_1| + |\overline{x}_0\overline{x}_1|,$$

we see that (1) and (2) must be equalities, and therefore

$$\gamma_t[u_0, u_1], \qquad l \le t \le l + r, \tag{3}$$

are minimizing geodesics filling a flat totally geodesic rectangle.

Now for $\delta \to 0$, we obtain

$$\begin{aligned} f(r-\delta) &\ge |x_1 P(\sigma_{u_1}(l+r-\delta))| \ge |\overline{x}_0 x_1| - |\overline{x}_0 P(\sigma_{u_1}(l+r-\delta))| \\ &\ge |\overline{x}_0 x_1| - |\sigma_{u_1}(l+r-\delta)\sigma_{u_0}(l+r)| \\ &\ge |\overline{x}_0 x_1| - |\sigma_{u_1}(l+r)\sigma_{u_0}(l+r)| - O(\delta^2) \\ &= |\overline{x}_0 x_1| - |x_0 x_1| - O(\delta^2) = |\overline{x}_0 x_0| - O(\delta^2) = f(r) - O(\delta^2), \end{aligned}$$

where we have used the definition of $\overline{x}_0$ and distance nonincreasing property of P in the third inequality, and (3) in the fourth one.

Thus $f(r) \equiv 0$ for $0 \le r \le \varepsilon(l)$, and (A) is proved for $0 \le t \le l + \varepsilon(l)$. To prove (B) for such t one can repeat a part of the argument above, up to assertion (3), taking into account that (x_0, ν_0), γ, x_1 can now be chosen arbitrarily, and $\overline{x}_0 = x_0$, $\overline{x}_1 = x_1$.

Assertion (C) is an easy corollary of (A), (B) and the distance decreasing property of P. Indeed, let x be an interior point of a minimizing geodesic $\gamma \subset S$, σ be a normal geodesic starting at x. Then, according to (B), we can construct a flat totally geodesic strip spanned by γ and σ, and, for any point y on σ, say $y = \sigma(t)$, we can define a lift γ_y of γ through y as a horizontal geodesic γ_t of that strip. This lift is independent of σ: if incidentally $y = \sigma'(t')$, then the corresponding lift γ'_y must coincide with γ_y because otherwise $|\gamma'_y(u_0)\gamma_y(u_1)| < |\gamma(u_0)\gamma(u_1)|$, and this would contradict (A) and the distance decreasing property of P.

Thus we have correctly defined continuous horizontal distribution. Similar arguments show that P has a correctly defined differential—a linear map which is isometric on horizontal distribution and identically zero on its orthogonal complement. For example, suppose two geodesics $\gamma^1, \gamma^2 \subset S$ are orthogonal at their intersection point x. Then their lifts γ^1_y, γ^2_y are orthogonal at y, because otherwise we would have $|\gamma^1_y(u_0)z| < |\gamma^1(u_0)P(z)|$ for some point z on γ^2_y close to y.

The estimate on the second fundamental form of the fiber $P^{-1}(x)$ at y follows from the inequality $|P^{-1}(x)\gamma_y(u_0)| \ge |x\gamma(u_0)|$, valid for all minimizing geodesics $\gamma \subset S$ passing through x, and from the standard estimate of the second fundamental form of a metric sphere in nonnegatively curved manifold.

Remarks. (1) The fibers of the submersion P are not necessarily isometric to each other, and not necessarily totally geodesic (see [6]).

(2) Existence of a Riemannian submersion of M onto S was conjectured some time ago by D. Gromoll.

(3) It would be interesting to find a version of our theorem for Alexandrov spaces (which may occur, for instance, as Gromov–Hausdorff limits of blowups of Riemannian manifolds, collapsing with lower bound on sectional curvature). We hope to address this and other rigidity problems for Alexandrov spaces elsewhere.

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